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TWO-VELOCITY HYDRODYNAMICS AND THERMODYNAMICS

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The equations of two-velocity hydrodynamics are obtained on the basis of general principles — thermodynamics, Galilean invariance, and the laws of conservation of energy, mass, momentum, etc. In a conceptual sense, the article develops previous work done by the author [1-3]. It must be emphasized that thermodynamics should also be regarded as two-velocity in the present investigation. The notion of two-velocity thermodynamics first appeared in Landau's theory of superfluidity [4], although there was no followup to this in the literature. Only recently were Landau's ideas used to derive the equations of two-velocity hydrodynamics in porous, elastically deformable media [5, 6].

The equations are constructed in two stages. First-order differential equations are found in the first stage (the stage described in this article). With a certain degree of conditionality, these expressions can be referred to as equations of "ideal two-velocity hydrodynamics." In the second stage, the equations are supplemented by "diffusion" terms containing second derivatives (this will be the subject of the next article). In both stages, the central role is played by the tensor classification of the quantities that characterize the state of the system and the corresponding tensor classification of the basic equations

1. Tensor Classification. Regarding the system that is of interest to us, let the state of the system at the point $x \equiv (x^\alpha) \equiv (x^0, x^1, \ldots, x^n)$ $(x^0 = t, n > 1)$ be characterized by the set of quantities $u(x) \equiv (u_1(x), \ldots, u_m(x))$. We introduce the Galilean group Γ, which includes the following coordinate transformations: time shifts and translations, spatial rotations (with a determinant of +1) forming the group SO(n), Galilean transformations, and inversions. A corresponding transformation $u \to \tilde{u}$ is specified for each transformation of coordinates $x \to \tilde{x}$ from Γ. We will later need to find a general form for quantities associated with a certain type of tensor, along with a specific structure (the problem of tensor classification). The criterion we will establish below will prove useful in this regard.

Let there be a group G of transformations $\tilde{x} = \tilde{x}(x)$, $\tilde{u} = \tilde{u}(u, x)$.

Definition 1.1. We will refer to the set of quantities $T(x, u) \equiv (T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}(x, u))$ of type (p, q) if for any transformation $(x, u) \to (\tilde{x}, \tilde{u})$ from G

$$\tilde{T}^{\alpha_1 \ldots}_{\beta_1 \ldots \beta_q \alpha_1} \equiv \frac{\partial \tilde{x}^{\alpha_1}}{\partial x^{\gamma_1}} \ldots \frac{\partial \tilde{x}^{\alpha_p}}{\partial x^{\gamma_p}} \frac{\partial x^{\delta_1}}{\partial \tilde{x}^{\beta_1}} \ldots \frac{\partial x^{\delta_q}}{\partial \tilde{x}^{\beta_q}} T^{\gamma_1 \ldots \gamma_p}_{\delta_1 \ldots \delta_q}(x, u) = T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}(\tilde{x}, \tilde{u}). \tag{1.1}$$

In particular, in accordance with (1.1), T(x, u) is a scalar if $T(x, u) = (T(x, \tilde{u})$, i.e. if T is invariant to G. Here and below, we adopt the usual tensor rule of summation over repeating tensorial indices. The Greek-letter indices take values of 0, 1,..., n, the Roman-letter indices values of 1,..., n.

Let G be a one-parameter Lie group of transformations $\tilde{x} = \tilde{x}(x; \varepsilon)$, $\tilde{u} = \tilde{u}(x, u; \varepsilon)$, where ε is the group parameter. We find the necessary criterion by using standard group methods [7, 8].

Proposition 1.1. *For T(x, u) to be a G-tensor of the type (p, q), it is necessary and sufficient that*

$$XT^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \frac{\partial \xi^{\alpha_1}}{\partial x^{\lambda_1}} T^{\lambda_1 \alpha_2 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} + \ldots + \frac{\partial \xi^{\alpha_p}}{\partial x^{\lambda_p}} T^{\alpha_1 \ldots \alpha_{p-1} \lambda_p}_{\beta_1 \ldots \beta_q} - \frac{\partial \xi^{\mu_1}}{\partial x^{\beta_1}} T^{\alpha_1 \ldots \alpha_p}_{\mu_1 \beta_2 \ldots \beta_q} - \frac{\partial \xi^{\mu_q}}{\partial x^{\beta_q}} T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_{q-1} \mu_q}, \tag{1.2}$$

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where X is an operator of the group G,

$$X \equiv \xi^{\alpha}(x)\frac{\partial}{\partial x^{\alpha}} + \eta^{s}(x, u)\frac{\partial}{\partial u^{s}} \quad (s = 1,..., m).$$

If T is a scalar, then (1.2) is transformed into the invariance condition XT = 0. If T depends on (x, u, $\partial u/\partial x$), then definition 1.1 obviously changes form, while in (1.2) X is replaced by the first continuation $pr^{(1)}X$, etc.

2. One-Velocity Hydrodynamics. Let us examine the case of one-velocity hydrodynamics for the sake of illustration. Let the state of the system be defined by the set $u \equiv (\rho, s, v)$ (where $\rho > 0$ is density, s is entropy — a Galilean scalar, and $v \equiv (v^1, v^2, v^3)$ is the velocity vector).

We assume that the thermodynamics is given, i.e. we determine internal energy $\varepsilon(\rho, s)$. The following equality is valid for the latter

$$d\varepsilon = -pd(1/\rho) + Tds \tag{2.1}$$

(p is pressure and T is absolute temperature). The energy of a unit volume $E = \rho(\varepsilon + v^2/2)$.

We further assume that the dynamics of the system is described by a first-order quasilinear system

$$\frac{\partial u}{\partial t} + C^{k}(u)\frac{\partial u}{\partial x^{k}} = F(u), \tag{2.2}$$

which is invariant relative to Γ. For the sake of simplicity, we assume that F = 0. The requirement of invariance causes Eqs. (2.2) to take the form [1, 9]:

$$\frac{\partial \rho}{\partial t} + v^{k}\frac{\partial \rho}{\partial x^{k}} + \alpha(\rho, s)\frac{\partial v^{k}}{\partial x^{k}} = 0; \tag{2.3a}$$

$$\frac{\partial s}{\partial t} + v^{k}\frac{\partial s}{\partial x^{k}} + \beta(\rho, s)\frac{\partial v^{k}}{\partial x^{k}} = 0; \tag{2.3b}$$

$$\frac{\partial v^{j}}{\partial t} + v^{k}\frac{\partial v^{j}}{\partial x^{k}} + A(\rho, s)\frac{\partial \rho}{\partial x^{j}} + B(\rho, s)\frac{\partial s}{\partial x^{j}} = 0. \tag{2.3c}$$

Here, α, β, A, and B are arbitrary smooth functions.

The third hypothesis states that the exact law of conservation of energy is a consequence of (2.3)

$$\frac{\partial E(u)}{\partial t} + \frac{\partial R^{k}(u)}{\partial x^{k}} = 0, \tag{2.4}$$

where the fluxes $R^k(u)$ must be determined. After calculations are performed, it follows from the above hypotheses that $\alpha = \rho$ and that (2.3a), (2.3b) can be represented in divergent form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^{k}}(\rho v^{k}) = 0; \tag{2.5a}$$

$$\frac{\partial}{\partial t}(\rho v^{j}) + \frac{\partial}{\partial x^{k}}(\rho v^{j}v^{k} + G\delta^{jk}) = 0, \; G \equiv p + \rho T\beta \tag{2.5b}$$

(δ^{ik} is the Kronecker symbol). Here, the energy conservation law (2.4) has the form

$$\frac{\partial}{\partial t}[\rho(\varepsilon + v^2/2)] + \frac{\partial}{\partial x^{k}}[\rho v^{k}(\varepsilon + v^2/2 + G/\rho)] = 0. \tag{2.6}$$

With the use of (2.5a), Eq. (2.3b) is also reduced to divergent form (at least locally)

$$\frac{\partial}{\partial t}(\rho\bar{s}) + \frac{\partial}{\partial x^{k}}(\rho\bar{s}v^{k}) = 0, \; \bar{s} = \bar{s}(\rho, s). \tag{2.7}$$

For this to occur, $\bar{s}$ must satisfy the equation

$$\frac{\partial \bar{s}}{\partial \rho} + (\beta/\rho)\frac{\partial \bar{s}}{\partial s} = 0. \tag{2.8}$$

We supplement (2.8) with the initial conditions $(\rho = \rho_0) \Rightarrow (\tilde{s} = \tilde{s}_0(s))$. We choose $\tilde{s}_0$ so that $\partial\tilde{s}_0/\partial s \neq 0$. Equation (2.8) then has a smooth solution for which (at least locally) $\partial\tilde{s}/\partial s \neq 0$. We change over from the variables (ρ, s) to $(\rho, \tilde{s})$. The Jacobian of the transformation $(\rho, s) \to (\rho, \tilde{s})$ is nontrivial, so this transformation is one-to-one locally. Differential form (2.1) is now written as:

$$d\varepsilon = -\tilde{p}d(1/\rho) + Td\tilde{s},\ \tilde{p} = p + \rho T\beta = G,\ T = \tilde{T}\partial\tilde{s}/\partial s. \tag{2.9}$$

Equations (2.5-2.7), (2.9) have the same form as the equations of gas dynamics, but (if $\beta \neq 0$) T and $\tilde{p}$ do not have the physical meanings of temperature and pressure. We therefore need one more hypothesis — a hypothesis that will lead to $\beta = 0$. As an example, we can take the law of conservation of entropy

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\rho s v^k) = 0. \tag{2.10}$$

Thus, the correct laws of conservation of mass and momentum follow from thermodynamics, energy conservation law (2.4) (accurate to within the fluxes R^k), scalar conservation law (2.10), the requirement of Galilean invariance, and structural hypothesis (2.2).

The result is easily generalized to a system whose state is determined by the set $u = (\rho, s, c, v)$, $c = (c_1 \ldots, c_\nu)$, where c_α are scalars. For an unambiguous definition, in addition to (2.10) we need to establish ν independent scalar conservation laws

$$\frac{\partial}{\partial t}[\rho\lambda_\alpha(\rho, s, c)] + \frac{\partial}{\partial x^k}[\rho\lambda_\alpha(\rho, s, c)v^k] = g_\alpha(\rho, s, c),\ \alpha = 1, \ldots, \nu$$

or to adopt certain analogs of these hypotheses. If $g_\alpha \neq 0$, then $\theta(\rho, s, c)$ appears in the right side of (2.4) in place of 0.

3. One-Velocity Hydrodynamics. Another Approach. The procedures used in Part 2 make it possible to construct general equations for different physical systems that meet two requirements. First, there is a suitable analog of the Gibbs equation (2.1). Secondly, a law has been established for transforming the quantities $(u_0, \ldots u_m)$ in each transformation from Γ. However, the complexity of the calculations increases sharply for two-velocity and more general systems. We will therefore describe a simpler approach which makes use of information on the structure of the sought equations (mainly the fact that they are divergent and are classified according to type of tensor).

Let the state of the system again be determined by the set $u \equiv (\rho, s, v)$. We assume that $V = (V^\alpha) \equiv (1, v) = (1, v^1, v^2, v^3)$. The first step entails solving the problem of tensor classification. This step is simple for the case we are considering.

Proposition 3.1.

1. The quantity J(u) is a G-scalar (Galilean scalar) when and only when $J = J(\rho, s)$.

2. The set of values $X(u) \equiv (X^\alpha(u))$ is a Γ-vector (Galilean vector) when and only when $X = a(\rho, s)V$, where a is a scalar.

3. The set of values $T(u) \equiv (T^{\alpha\beta}(u))$ is a Γ-tensor of the type (2, 0) (Galilean tensor) when and only when $T = A(\rho, s)V \otimes V + B(\rho, s)\delta$, where A and B are scalars,

$$\delta \equiv \begin{bmatrix} 0 & 0 \\ 0 & I_3 \end{bmatrix},\ I_3 \text{ — is a three-dimensional unit.}$$

Definition 3.1.

1. The quantity $J(u, \partial u/\partial x)$ is called a divergent first-order Γ-scalar if J is a Γ-scalar and has a divergent structure: $J = \partial/\partial x^\alpha[T^{\alpha\beta}(u)]$ or $X = \mathrm{div}T$.

2. The set of values $X(u, \partial u/\partial x) \equiv (X^\beta(u, \partial u/\partial x))$ is called a divergent first-order Γ-vector if X is a Γ-vector and $x^\beta = \partial/\partial x^\alpha[T^{\alpha\beta}(u)]$ or $X = \mathrm{div}T$.

Similarly, $J(u, \partial u/\partial x, \ldots, \partial^m u/\partial x^m)$ is a divergent Γ-scalar of order m if J is a Γ-scalar and $J = \mathrm{div}X$, $X = (X^\alpha)$, $X^\alpha = X^\alpha(u, \partial u/\partial x, \ldots, \partial^{m-1} u/\partial x^{m-1})$, etc.

Proposition 3.2. *1. The quantity J is a divergent first-order Γ-scalar when and only when $J = \mathrm{div}X(u)$ and $X(u)$ is a Galilean vector, $X(u) = a(\rho, s)V$.*

2. The set of values $X = (X^\alpha)$ forms a divergent first-order Γ-vector when and only when $X = divT$, where $T = A(\rho, s)V \otimes V + B(\rho, s)\delta$.

Thus, the state of the system is given by one base scalar s and one Galilean base vector ρV (**mass-momentum vector**). We similarly classify the basic equations. Specifically, they consist of conservation laws of the following types — one scalar law (**entropy conservation law**) and one vector law (**mass-momentum conservation law**).

By a conservation law, we mean an expression of the form

$$\frac{\partial \varphi^\alpha(u)}{\partial x^\alpha} = f(u) \tag{3.1}$$

(the law of conservation of angular momentum is a natural exception, although it is not written out in explicit form). If f = 0, then conservation law (3.1) will be considered exact. As in Part 2, we will simplify the problem by using only exact conservation laws in this section.

In accordance with proposition 3.2, vector conservation law is as follows: $\partial T^{\alpha\beta}/\partial x^\alpha = 0$, $T = A(\rho, s)\, V \otimes V + G(\rho, s)\delta$. When $\beta = 0$, the equation must be the mass conservation law, i.e. have the form $\partial\rho/\partial x + \partial(\ldots)^k/\partial x^k = 0$. Thus, A = ρ, and the vector equation takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k) = 0; \tag{3.2a}$$

$$\frac{\partial(\rho v^j)}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^j v^k + G(\rho, s)\delta^{jk}) = 0. \tag{3.2b}$$

The scalar equation must be the entropy conservation law, i.e. must have the form $d(\rho s)/\partial t + \partial(\ldots)^k/\partial x^k = 0$. Thus, in accordance with proposition (3.2),

$$\frac{\partial(\rho s)}{\partial t} + \frac{\partial(\rho s v^k)}{\partial x^k} = 0. \tag{3.3}$$

As in Part 2, Eqs. (3.2)-(3.3) lead to the exact energy conservation law

$$\frac{\partial}{\partial t}[\rho(\varepsilon + v^2/2)] + \frac{\partial R^k(u)}{\partial x^k} = 0, \tag{3.4}$$

where (2.1) must be satisfied for $\varepsilon(\rho, s)$. We transform differential equation (2.1) as follows. First we write it in the form

$$d(\rho\varepsilon) = (\varepsilon + p/\rho - Ts)d\rho + Td(\rho s). \tag{3.5}$$

As can be seen from a comparison of the structure of Eqs. (3.2)-(3.4) and (3.5), differential equation (3.5) is internally connected with system (3.2)-(3.4). We will therefore refer to (3.5) as the first **fundamental (thermodynamic) differential form**. The quantities (ρ, ρs) will be called **canonical (thermodynamic) variables of the first kind**, the function $\rho\varepsilon = \Phi(\rho, \rho s)$ will be called the **first canonical (thermodynamic) potential**, the quantities (γ, T) ($\gamma \equiv \varepsilon + p/\rho - Ts$ — the specific Gibbs potential) will be referred to as **conjugate (thermodynamic) variables or canonical (thermodynamic) variables of the second kind**, and the transformation (ρ, ρs) → (γ, T) will be called a **canonical transformation**. If it is locally one-to-one (as it will be if, for example, the potential $\Phi(\rho, \rho s)$ is strictly convex), then we will also have the inverse transformation (γ, T) → (ρ, ρs). In this case, we can use a Legendre transformation to change over from the potential $\Phi(\rho, \rho s)$ to the potential $\Phi_*(\gamma, T) \equiv \gamma\rho + T\rho s - \Phi$. We find from this that $\Phi_* = p$. We will refer to thermodynamic potential p(γ, T) as a **conjugate potential or potential of the second kind.** It follows from (3.5) that

$$dp = \rho d\gamma + \rho s dT. \tag{3.6}$$

We now change from form (3.5) to the **complete differential form** (compare with (3.2)-(3.4))

$$dE \equiv d[\rho(\varepsilon + v^2/2)] = q_0 d\rho + q_j d(\rho v^j) + q_4 d(\rho s). \tag{3.7}$$

Here, E is regarded as a function of the variables (ρ, ρv, ρs). Expressions (3.5) and (3.7) are equivalent if

$$q_0 = \gamma - v^2/2,\ q_j = v^j,\ q_4 = T.$$

We call the set of values $q = (q_0,\ldots q_4)$ **integrating factors.** If the transformation $(\rho, \rho s) \to (\gamma, T)$ is locally one-to-one, then the transformation $(\rho, \rho v, \rho s) \to q$ will also be so, and system of conservation laws (3.2)-(3.3) will be complete in the sense of the term employed in [2]. Here, conservation laws (3.2)-(3.3) are regarded as fundamental laws, while (3.4) is regarded as a closing law (which is permissible in the given case because we are interested only in smooth solutions). In accordance with (3.7),

$$\frac{\partial E}{\partial \rho} = \gamma - v^2/2,\ \frac{\partial E}{\partial(\rho v^j)} = v^j,\ \frac{\partial E}{\partial(\rho s)} = T.$$

We multiply (3.2a) by q_0, (3.26) by q_j, and (3.3) by q_4 and require that we obtain (3.4) as a result. After completing the calculations, we find from this that $G = p$, i.e. system (3.2)-(3.3) agrees with the classical system of equations in gas dynamics.

A similar approach can be used for more complex systems.

4. Two-Velocity Mechanical System. Let the local state of the system be determined by the set $(\rho_{(1)}, v_{(1)}, \rho_{(2)}, v_{(2)})$, where $\rho_{(i)}(x) \geq 0$ is the density of the i-th component, $\rho_{(1)} + \rho_{(2)} > 0$; $v_{(i)} \equiv (v_{(i)}{}^1,\ldots, v_{(i)}{}^n)$, $n > 1$ is the velocity of the i-th component. It is convenient to change over to the quantities

$$\rho \equiv \rho_{(1)} + \rho_{(2)},\ \varkappa \equiv \rho_{(1)}/\rho,\ v \equiv \varkappa w_{(1)} + (1 - \varkappa)v_{(2)},$$
$$w = v_{(1)} - v_{(2)}. \tag{4.1}$$

Now we can assume that the state of the system is determined by the set $u \equiv (\rho, \varkappa, v, w)$. We find from (4.1) that $v_{(1)} = v + (1 - \varkappa)w$, $v_{(2)} = v - \varkappa w$. We introduce Galilean vectors

$$V = (V^\alpha) \equiv (1, v),\ W = (W^\alpha) \equiv (0, w).$$

Let us proceed to the tensor classification.

Proposition 4.1.

1. The quantity $J(u)$ is a Γ-scalar when and only when $J = J(\rho, \varkappa, w^2)$.

2. The set of values $X(u) = (X^\alpha(u))$ is a Galilean vector when and only when $X = aV = bW$.

3. The set of values $T(u) = (T^{\alpha\beta}(u))$ is a Galilean tensor of the type (2, 0) when and only when $T = AV \otimes V + BV \otimes W + CW \otimes V + DW \otimes W + E\delta$. Here, a, b, A, B, C, D, and E are scalars and thus depend only on ρ, $\varkappa$, and w^2.

Thus, the initial set of scalars $(\rho, \varkappa)$ is supplemented by an independent scalar w^2, which satisfies the conditions necessary for the development of two-velocity thermodynamics.

Proposition 4.2.

1. The quantity $J(u, \partial u/\partial x)$ is a divergent Galilean scalar when and only when $J = divX(u)$ and $X(u)$ is a Galilean vector.

2. The set of values $X(u, \partial u/\partial x) = (X^\alpha)$ forms a divergent Galilean vector when and only when $X = divT(u)$ and $T(u)$ is a Galilean tensor of the type (2, 0).

Now we need to introduce the differential form of energy. Various hypotheses are possible here, and they will be formulated in the order of their generality.

Hypothesis I. We assume that the kinetic energy of the system is equal to the sum of the kinetic energies of the components, i.e. the total energy of a unit volume is

$$E(u) = \rho_{(1)}v_{(1)}^2/2 + \rho_{(2)}v_{(2)}^2/2 + \rho\varepsilon_0(\rho, \varkappa).$$

Here, ε_0 is the internal energy with stationary components. Thus

$$d\varepsilon_0 = -pd(1/\rho) + \mu_{(1)0}d\varkappa\,. \tag{4.2}$$

Taking (4.1) into account, we find that

$$E = \rho(\varepsilon + v^2/2),\ \varepsilon \equiv \varepsilon_0 + \varkappa(1 - \varkappa)w^2/2. \tag{4.3}$$

It can be seen from (4.3) that $\varepsilon(\rho, \varkappa, w^2)$ plays the role of internal energy. It follows from (4.2), (4.3) that

$$d\varepsilon = -pd(1/\rho) + [\mu_{(1)0} + (1 - 2\varkappa)w^2/2]d\varkappa + \varkappa(1 - \varkappa)w^k dw^k. \tag{4.4}$$

Thus, in the final analysis, hypothesis I states that $E \equiv \rho(\varepsilon + v^2/2)$. Meanwhile, (4.4) is satisfied for ε.

Hypothesis II. This hypothesis is a generalization of hypothesis I. In it, we assume that $E = \rho[\varepsilon(\rho, \varkappa, w^2) + v^2/2]$. Meanwhile, an expression which is more general than (4.4) is used for $d\varepsilon$

$$d\varepsilon = -p'd(1/\rho) + \mu'_{(1)}d\varkappa + j^k dw^k,$$

where $p' = p'(\rho, \varkappa, w^2)$, etc.

Hypothesis III. Here, we take the general relation $E = E(\rho, \varkappa, v^2, w^2, (v \cdot w))$, which follows from the requirement that E be invariant relative to the group of translations and rotations; $(v \cdot w)$ is the scalar product of the vectors $v, w \in R^3$.

Hypothesis I may be approximate, but it is also simple and clear — which makes it a reasonable starting point for constructing a theory of two-velocity systems. We will therefore take it to be valid in our subsequent discussions.

Hypothesis II makes it possible to refine hypothesis I. For example, let us assume that component 2 is composed of fine particles moving in a carrier fluid with the relative velocity w. Each particle creates perturbations in the fluid, these perturbations each contributing to the kinetic energy. If the phenomenological state of the system can be described satisfactorily by the set $u = (\rho, \varkappa, v, w)$ (and, if not, then the initial set u should be expanded), then we obtain some variant of hypothesis II that differs from I.

For Galilean systems, hypothesis III reduces to hypothesis II. Thus, in two-velocity mechanics, hypothesis II is the more general proposition.

The state of the system is determined by two Galilean vectors, such as ρV and $\rho_{(1)}V_{(1)}$. We thus assume that the equations of motion decompose into two groups, each of which forms a Galilean vector. The first group determines the mass-momentum conservation law for the system as a whole, while the second gives the mass-momentum equation for component 1.

The mass-momentum conservation law for the system must be exact. It should therefore have the form $\mathrm{div}T = 0$, where the structure of the tensor is determined by proposition 4.1. For the law of conservation of angular momentum to also be satisfied by the system as a whole, tensor T should be symmetric. The first equation ($\beta = 0$) should be the mass conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^k)}{\partial x^k} = 0. \tag{4.5a}$$

Thus, $T = \rho V \otimes V + \lambda(\rho, \varkappa, w^2)W \otimes W + G(\rho, \varkappa, w^2)\delta$. We then have

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^k}(\rho v^i v^k + \lambda w^i w^k + G\delta^{ik}) = 0. \tag{4.5b}$$

The mass-momentum balance equation for component 1 should be a Galilean vector and have the structure of a first-order quasilinear system. Isolating the divergent part of the system, we write it as follows: $\mathrm{div}T_{(1)}(u) = F_{(1)}(u, \partial u/\partial x)$ ($T_{(1)}$ is a tensor and $F_{(1)}$ is a vector). In accordance with our proposition, the force $F_{(1)}$ acting on the component should be independent of $\partial u/\partial t$, i.e. its structure should be as follows:

$$F_{(1)} = A^k_{(1)}(u)\partial u/\partial x^k + B_{(1)}(u). \tag{4.6}$$

We represent the general form of a Galilean vector having the structure (4.6) as

$$\begin{aligned} F_{(1)} = {} & q_{(1)}V + f_{(1)}W + a_{(1)}(\delta \cdot \nabla\rho) + b_{(1)}(\delta \cdot \nabla\varkappa) + d_{(1)}(\delta \cdot \nabla w^2) \\ & + A_{(1)}(W \cdot \nabla\rho)V + B_{(1)}(W \cdot \nabla\varkappa)V + D_{(1)}(W \cdot \nabla w^2)V \\ & + L_{(1)}(W \cdot \nabla\rho)W + M_{(1)}(W \cdot \nabla\varkappa)W + K_{(1)}(W \cdot \nabla w^2)W \\ & + \xi_{(1)}(\partial V/\partial x \cdot W) + \eta_{(1)}(\partial W/\partial x \cdot W) \\ & + \varphi_{(1)}(\mathrm{div}V)V + \psi_{(1)}(\mathrm{div}W)V + \alpha_{(1)}(\mathrm{div}V)W + \beta_{(1)}(\mathrm{div}W)W, \end{aligned} \tag{4.7}$$

where the coefficients $q_{(1)},\ldots, \beta_{(1)}$ depend only on ρ, $\varkappa$, and w^2;

$$(\delta\cdot\nabla\rho)^\beta \equiv \delta^{\alpha\beta}\partial\rho/x^\alpha;\ (W\cdot\nabla\rho) \equiv W^\alpha\partial\rho/x^\alpha;\ (\partial V/\partial x\cdot W)^\beta \equiv \frac{\partial V^\beta}{\partial x^\alpha}W^\alpha$$

etc. The first equation ($\beta = 0$) should be the mass conservation law for component 1. It is written in the form

$$\frac{\partial}{\partial t}(\rho_{(1)}) + \frac{\partial}{\partial x^k}(\rho_{(1)}v_{(1)}) = \tilde{q}_{(1)}(u)$$

($\tilde{q}_{(1)}(u)$ is a scalar, which means that $\tilde{q}_{(1)} = \tilde{q}_{(1)}(\rho, \kappa, w^2)$). With allowance for (4.1), this equation is written as follows:

$$\frac{\partial}{\partial t}(\varkappa\rho) + \frac{\partial}{\partial x^k}[\varkappa\rho v^k + \varkappa(1-\varkappa)\rho w^k] = \tilde{q}_{(1)}(\rho, \varkappa, w^2). \tag{4.8a}$$

We thus take $q_{(1)} = \tilde{q}_{(1)}$, $A_{(1)} = B_{(1)} = D_{(1)} = \varphi_{(1)} = \psi_{(1)} = 0$. It seems reasonable to take the following as the momentum balance equation for component 1 when hypothesis I is adopted for energy

$$\frac{\partial}{\partial t}(\rho_{(1)}v^j_{(1)}) + \frac{\partial}{\partial x^k}(\ldots)^{kj} = F^j_{(1)}. \tag{4.9}$$

In accordance (4.8a), (4.9), the general form of tensor $T_{(1)}$ is:

$$T_{(1)} = \varkappa\rho V\otimes V + \varkappa(1-\varkappa)\rho(V\otimes W + W\otimes V) + \lambda_{(1)}W\otimes W + G_{(1)}\delta.$$

Here, $\lambda_{(1)} = \lambda_{(1)}(\rho, \varkappa, w^2)$; $G_{(1)} = G_{(1)}(\rho, \varkappa, w^2)$. We thus arrive at the momentum balance equation for component 1:

$$\frac{\partial}{\partial t}[\varkappa\rho v^k + \varkappa(1-\varkappa)\rho w^j] + \frac{\partial}{\partial x^k}T^{kj}_{(1)} = F^j_1. \tag{4.8b}$$

With allowance for (4.3), (4.4), one consequence of (4.5) and (4.8) is the energy conservation law for the system as a whole. Since $F_{(1)}$ does not contain any terms that have derivatives, we write this law in the form

$$\frac{\partial E}{\partial t} + \frac{\partial R^k(u)}{\partial x^k} = \theta(u). \tag{4.10}$$

We thus find the structure of equations (4.5), (4.8) and closing conservation law (4.10). The explicit form of the terms having a derivative with respect to t is already determined, but it is still necessary to establish the expression for the fluxes. The problem subsequently reduces essentially to calculation of the integrating factors. As in Part 3, we transform (4.4) on the basis of the structure of the basic equations:

$$\begin{gathered} d(\rho\varepsilon) = \gamma d\rho + \mu_{(1)}d(\rho\varkappa) + w^k d[\varkappa(1-\varkappa)\rho w^k], \\ \gamma \equiv \gamma_0 - \varkappa^2w^2/2,\ \gamma_0 \equiv \varepsilon_0 + p/\rho - \varkappa\mu_{(1)0},\ \mu_{(1)} \equiv \mu_{(1)0} - (1-2\varkappa)w^2/2. \end{gathered} \tag{4.11}$$

From here, we easily find the complete differential form

$$dE = d[\rho(\varepsilon + v^2/2)] = q_0 d\rho + q_j d(\rho v^j) + q_{n+1}d(\varkappa\rho) + q_{n+1+j}d[\varkappa\rho v^j + \varkappa(1-\varkappa)\rho w^j]; \tag{4.12}$$

$$q_0 = \gamma_0 - \varkappa^2w^2/2 + \varkappa v^k w^k - v^2/2 = \gamma_0 - v^2_{(2)}/2; \tag{4.13a}$$

$$q_j = v^j - \varkappa w^j = v^j_{(2)}; \tag{4.13b}$$

$$q_{n+1} = \mu_{(1)0} - (1-2\varkappa)w^2/2 - v^k w^k = \mu_{(1)0} - v^2_{(1)}/2 + v^2_{(2)}/2; \tag{4.13c}$$

$$q_{n+1+j} = w^j. \tag{4.13d}$$

We multiply (4.5a) by q_0, (4.5b) by q_j, (4.8a) by q_{n+1}, and (4.8b) by q_{n+1+j} and require that the resulting expression agree with (4.10). After some lengthy but simple calculations, we arrive at the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^k)}{\partial x^k} = 0; \tag{4.14a}$$

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^k}[\rho v^i v^k + \varkappa(1-\varkappa)\rho w^i w^k + (\xi w^i w^k + G\delta^{ik})] = 0; \tag{4.14b}$$

$$G \equiv p + \alpha w^2; \tag{4.15}$$

$$\frac{\partial}{\partial t}(\rho\varkappa) + \frac{\partial}{\partial x^k}[\varkappa\rho v^k + \varkappa(1-\varkappa)\rho w^k] = q_{(1)}; \tag{4.16a}$$

$$\frac{\partial}{\partial t}[\varkappa\rho v^i + \varkappa(1-\varkappa)\rho w^i] \tag{4.16b}$$

$$+ \frac{\partial}{\partial x^k}[\varkappa\rho v^i v^k + \varkappa(1-\varkappa)\rho(v^i w^k + w^i v^k) + \varkappa(1-\varkappa)^2\rho w^i w^k + \tilde{\lambda}_{(1)} w^i w^k]$$

$$+ \varkappa\frac{\partial}{\partial x^k}(\xi w^i w^k + G\delta^{ik}) + \varkappa(1-\varkappa)\rho\frac{\partial \mu_{(1)0}}{\partial x^i} + \xi w^k\frac{\partial v^i}{\partial x^k} + \alpha w^i\frac{\partial w^k}{\partial x^k} = q_{(1)}v^i - fw^i + Q^i_{(1)},$$

$$Q^i_{(1)} \equiv \tilde{L}_{(1)}(w^i w^k - w^2\delta^{ik})\frac{\partial \rho}{\partial x^k} + \tilde{M}_{(1)}(w^i w^k - w^2\delta^{ik})\frac{\partial \varkappa}{\partial x^k} + \tilde{K}_{(1)}(w^i w^k - w^2\delta^{ik})\frac{\partial w^2}{\partial x^k} + \tilde{\eta}(w^k\frac{\partial w^i}{\partial x^k} - w^k\frac{\partial w^k}{\partial x^k}).$$

Here, ξ, α,..., $\tilde{\eta}$ depend only on the scalars ρ, $\varkappa$, and w^2. We note that $w^j Q^j = 0$, i.e. $w \perp Q$. Thus, Q does not contribute to the energy conservation law. The latter has the form

$$\frac{\partial}{\partial t}[\rho(\varepsilon + v^2/2)] + \frac{\partial}{\partial x^k}\{\rho v^k(\varepsilon + v^2/2 + G/\rho) \tag{4.17}$$

$$+ \rho w^k[\varkappa(1-\varkappa)(v\cdot w) + (\xi/\rho)(v\cdot w) + \varkappa(1-\varkappa)\mu_{(1)}]$$

$$+ \varkappa(1-\varkappa)(1-2\varkappa)w^2/2 + (\tilde{\lambda}_{(1)}/\rho)w^2]\} = \theta;$$

$$\theta = [\mu_{(1)0} - (1-2\varkappa)w^2/2]q_{(1)} - fw^2. \tag{4.18}$$

If $w \approx 0$, then (4.14), (4.16a) change into the equations of one-velocity hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^k)}{\partial x^k} = 0; \tag{4.19a}$$

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^k}[\rho v^i v^k + p\delta^{ik}] = 0; \tag{4.19b}$$

$$\frac{\partial}{\partial t}(\varkappa\rho) + \frac{\partial}{\partial x^k}(\varkappa\rho v^k) = 0. \tag{4.20}$$

Here, integrating factors (4.13a-4.13c) become the integrating factors for system (4.19-4.20) $q_0 = \gamma_0 - v^2/2$, $q_j = v^j$, $q_{n+1} = \mu_{(1)0}$. We thus make a proper transition to the limit of one-velocity hydrodynamics, which in principle allows us to construct solutions for two-velocity hydrodynamics in the form of series in powers of w^2 by using known solutions of one-velocity hydrodynamics as the zeroth approximation.

5. Minimal System. In light of the complexity of Eqs. (4.14)-(4.18), it seems reasonable to first examine (and use in applications) their simplest variant. Then additional terms from (4.14)-(4.18) can be introduced into the system as the need arises. We therefore put $\xi = \alpha = \tilde{\lambda}_{(1)} = \tilde{L}_{(1)} = \tilde{M}_{(1)} = \tilde{K}_{(1)} = \tilde{\eta} = 0$, which leads to the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v^k}{\partial x^k} = 0; \tag{5.1a}$$

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^k}[\rho v^i v^k + \varkappa(1-\varkappa)\rho w^i w^k + p\delta^{ik}] = 0; \tag{5.1b}$$

$$\frac{\partial}{\partial t}(\varkappa\rho) + \frac{\partial}{\partial x^k}[\varkappa\rho v^k + \varkappa(1-\varkappa)\rho w^k] = q_{(1)}; \tag{5.2a}$$

$$\frac{\partial}{\partial t}[\varkappa\rho v^j + \varkappa(1-\varkappa)\rho w^j]$$
$$+ \frac{\partial}{\partial x^k}[\varkappa\rho v^j v^k + \varkappa(1-\varkappa)\rho(v^j w^k + w^j v^k) + \varkappa(1-\varkappa)^2\rho w^j w^k] \tag{5.2b}$$
$$+ \varkappa\frac{\partial p}{\partial x^j} + \varkappa(1-\varkappa)\rho\frac{\partial \mu_{(1)0}}{\partial x^j} = q_{(1)}v^j - fw^j.$$

The energy conservation law for (5.1)-(5.2) has the form

$$\frac{\partial}{\partial t}[\rho(\varepsilon + v^2/2)] + \frac{\partial}{\partial x^k}\{\rho v^k(\varepsilon + v^2/2 + p/\rho)$$
$$+ \rho w^k[\varkappa(1-\varkappa)(v\cdot w) + \varkappa(1-\varkappa)\mu_{(1)0} + \varkappa(1-\varkappa)(1-2\varkappa)w^2/2]\} = \theta, \tag{5.3}$$

where θ is determined by Eq. (4.18). These equations are obtained in a more symmetrical and perhaps clearer form if we return to the original variables ($\rho_{(1)}$, $v_{(1)}$, $\rho_{(2)}$, $v_{(2)}$) and write the mass-momentum balance equation for each component:

$$\frac{\partial}{\partial t}\rho_{(1)} + \frac{\partial}{\partial x^k}(\rho_{(1)}v^k_{(1)}) = q_{(1)}; \tag{5.4a}$$

$$\frac{\partial}{\partial t}(\rho_{(1)}v^j_{(1)}) + \frac{\partial}{\partial x^k}(\rho_{(1)}v^j_{(1)}v^k_{(1)}) \tag{5.4b}$$
$$+ \varkappa\frac{\partial p}{\partial x^j} + \varkappa(1-\varkappa)\rho\frac{\partial \mu_{(1)0}}{\partial x^j} = q_{(1)}v^j - f(v^j_{(1)} - v^j_{(1)});$$

$$\frac{\partial}{\partial t}\rho_{(2)} + \frac{\partial}{\partial x^k}(\rho_{(2)}v^k_{(2)}) = q_{(2)}; \tag{5.5a}$$

$$\frac{\partial}{\partial t}(\rho_{(2)}v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(2)}v^j_{(2)}v^k_{(2)}) \tag{5.5b}$$
$$+ (1-\varkappa)\frac{\partial p}{\partial x^j} + \varkappa(1-\varkappa)\rho\frac{\partial \mu_{(2)0}}{\partial x^j} = q_{(2)}v^j - f(v^j_{(2)} - v^j_{(2)}).$$

Here, $q_{(2)} \equiv -q_{(1)}$; $\mu_{(2)0} \equiv -\mu_{(1)0}$. The energy conservation law can be written in the form:

$$\frac{\partial}{\partial t}(\rho\varepsilon_0 + \rho_{(1)}v^2_{(1)}/2 + \rho_{(2)}v^2_{(2)}/2) \tag{5.6}$$
$$+ \frac{\partial}{\partial x^k}\{\rho_{(1)}v^k_{(1)}[\varepsilon_0 + v^2_{(1)}/2 + p/\rho + (1-\varkappa)\mu_{(1)0}]$$
$$+ \rho_{(2)}v^k_{(2)}[\varepsilon_0 + v^2_{(2)}/2 + p/\rho + \varkappa\mu_{(2)0}]\} = \theta.$$

Adding (5.4a) to (5.5a) and adding (5.4b) to (5.5b), we obtain the mass-momentum conservation law for the system as a whole:

$$\frac{\partial}{\partial t}(\rho_{(1)} + \rho_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)}v^k_{(1)} + \rho_{(2)}v^k_{(2)}) = 0; \tag{5.7a}$$

$$\frac{\partial}{\partial t}(\rho_{(1)}v^j_{(1)} + \rho_{(2)}v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)}v^j_{(1)}v^k_{(1)} + \rho_{(2)}v^j_{(2)}v^k_{(2)} + p\delta^{jk}) = 0. \tag{5.7b}$$

Equations (5.1) and (5.7) are equivalent. Equations (5.4) and (5.5) were constructed completely symmetrically. Specifically, (5.4) becomes (5.5) and (5.5) becomes (5.4) if we make the substitution ((1), (2), $\varkappa$) → ((2), (1), $1-\varkappa$).

6. Hydraulic Analogy. As is known, there is an analogy between the classical equations of gas dynamics and the theory of shallow water. A similar analogy exists between (5.4)-(5.5) and the equations of two-layer shallow water.

The top layer is assigned the index (1), the bottom (2). The thickness of the j-th layer will be designated as $h_{(j)}$ (total depth $H = h_{(1)} + h_{(2)}$), mean velocity as $v_{(j)}$, and density as $\rho_{(j)}$ ($\rho_{(j)} = \text{const} > 0$).

The equations of two-layer shallow water have the form [10]

$$\frac{\partial h_{(1)}}{\partial t} + \frac{\partial}{\partial x^k}(v^k_{(1)} h_{(1)}) = 0; \tag{6.1}$$

$$\frac{\partial v^j_{(1)}}{\partial t} + v^k_{(1)} \frac{\partial v^j_{(1)}}{\partial x^k} + \frac{\partial}{\partial x^j}[g(h_{(1)} + h_{(2)})] = 0; \tag{6.2}$$

$$\frac{\partial h_{(2)}}{\partial t} + \frac{\partial}{\partial x^k}(v^k_{(2)} h_{(2)}) = 0; \tag{6.3}$$

$$\frac{\partial v^j_{(2)}}{\partial t} + v^k_{(2)} \frac{\partial v_{(2)}}{\partial x^k} + \frac{\partial}{\partial x^j}[g(\lambda h_{(1)} + h_{(2)})] = 0, \tag{6.4}$$

$$\lambda \equiv \rho_{(1)}/\rho_{(2)}.$$

We use (6.1)-(6.4) to find the mass-momentum balance equations for each layer:

$$\frac{\partial}{\partial t}(\rho_{(1)} h_{(1)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} h_{(1)} v^k_{(1)}) = 0; \tag{6.5a}$$

$$\frac{\partial}{\partial t}(\rho_{(1)} h_{(1)} v^j_{(1)} + \frac{\partial}{\partial x^k}(\rho_{(1)} h_{(1)} v^j_{(1)} v^k_{(1)}) \tag{6.5b}$$

$$+ g\rho_{(1)} h_{(1)} \frac{\partial h_{(1)}}{\partial x^j} + g\rho_{(1)} h_{(1)} \frac{\partial h_{(2)}}{\partial x^j} = 0;$$

$$\frac{\partial}{\partial t}(\rho_{(2)} h_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(2)} h_{(2)} v^k_{(2)}) = 0; \tag{6.6a}$$

$$\frac{\partial}{\partial t}(\rho_{(2)} h_{(2)} v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(2)} h_{(2)} v^j_{(2)} v^k_{(2)}) \tag{6.6b}$$

$$+ g\lambda\rho_{(2)} h_{(2)} \frac{\partial h_{(1)}}{\partial x^j} + g\rho_{(2)} h_{(2)} \frac{\partial h_{(2)}}{\partial x^j} = 0.$$

We write the energy conservation law in the form

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho_{(1)} h_{(1)} v^2_{(1)}/2 + \rho_{(2)} h_{(2)} v^2_{(2)}/2 \\ &+ g\rho_{(1)} h^2_{(1)}/2 + g\rho_{(2)} h^2_{(2)}/2 + g\rho_{(1)} h_{(1)} h_{(2)}) \\ &+ \frac{\partial}{\partial x^k}\{\rho_{(1)} h_{(1)} v^k_{(1)}(v^2_{(1)}/2 + gh_{(1)} + gh_{(2)}) \\ &+ \rho_{(2)} h_{(2)} v^k_{(2)}(v^2_{(2)}/2 + g\lambda h_{(1)} + gh_{(2)})\} = 0. \end{aligned} \tag{6.7}$$

Thus,

$$\begin{aligned} E \equiv &\rho_{(1)} h_{(1)} v^2_{(1)}/2 + \rho_{(2)} h_{(2)} v^2_{(2)}/2 \\ &+ (g\rho_{(1)} h^2_{(1)}/2 + g\rho_{(2)} h^2_{(2)}/2 + g\rho_{(1)} h_{(1)} h_{(2)}). \end{aligned}$$

We take

$$\omega_{(j)} \equiv \rho_{(j)} h_{(j)}, \ \omega \equiv \omega_{(1)} + \omega_{(2)}, \ \varkappa \equiv \omega_{(1)}/\omega,$$
$$v \equiv \varkappa v_{(1)} + (1 - \varkappa) v_{(2)}, \ w \equiv v_{(1)} - v_{(2)}.$$

We now represent energy E in the form

$$E = \omega(\varepsilon + v^2/2), \ \varepsilon = \varepsilon_0 + \varkappa(1 - \varkappa)w^2/2, \ \varepsilon_0 = \frac{g\omega}{2\rho_{(1)}}[\lambda + \varkappa^2(1 - \lambda)].$$

By differentiating, we find from $d\varepsilon_0 = -pd(1/\omega) + \mu_{(1)0}{}^{d\varkappa}$ that

$$p/\omega = \varepsilon_0, \ \mu_{(1)0} = (g\omega/\rho_{(1)})\varkappa(1 - \lambda).$$

Now (6.5)-(6.6) can be written in a form analogous to (5.4)-(5.5), where we took $q_{(j)} = 0$, $f = 0$:

$$\frac{\partial \omega_{(1)}}{\partial t} + \frac{\partial}{\partial x^k}(\omega_{(1)} v^k_{(1)}) = 0; \tag{6.8a}$$

$$\frac{\partial}{\partial t}(\omega_{(1)} v_{(1)}) + \frac{\partial}{\partial x^k}(\omega_{(1)} v^j_{(1)} v^k_{(1)})) + \varkappa \frac{\partial p}{\partial x^j} + \varkappa(1 - \varkappa)\omega \frac{\partial \mu_{(1)0}}{\partial x^j} = 0; \tag{6.8b}$$

$$\frac{\partial \omega_{(2)}}{\partial t} + \frac{\partial}{\partial x^k}(\omega_{(2)} v^k_{(2)}) = 0; \tag{6.9a}$$

$$\frac{\partial}{\partial t}(\omega_{(2)} v_{(2)}) + \frac{\partial}{\partial x^k}(\omega_{(2)} v^j_{(2)} v^k_{(2)}) + (1 - \varkappa)\frac{\partial p}{\partial x^j} + \varkappa(1 - \varkappa)\omega \frac{\partial \mu_{(2)0}}{\partial x^j} = 0, \mu_{(2)0} \equiv -\mu_{(1)0}. \tag{6.9b}$$

Let us consider the case of plane flow (n = 2). We find all of the conservation laws, which have the form

$$\frac{\partial \varphi(u)}{\partial t} + \frac{\partial \psi^k(u)}{\partial x^k} = 0, \; u \equiv (h_{(1)}, v_{(1)}, h_{(2)}, v_{(2)}), \quad v_{(j)} \equiv (v^1_{(j)}, v^2_{(j)}). \tag{6.10}$$

Three independent conservation laws have already been written: (6.1), (6.3), and (6.7). We still have to establish the law for the conservation of total momentum. This can be found by adding (6.5b) and (6.6b):

$$\frac{\partial}{\partial t}(\rho_{(1)} h_{(1)} v^j_{(1)} + \rho_{(2)} h_{(2)} v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} h_{(1)} v^j_{(1)} v^k_{(1)} + \rho_{(2)} h_{(2)} v^j_{(2)} v^k_{(2)} + g\rho_{(1)} h^2_{(1)}/2 + g\rho_{(2)} h^2_{(2)}/2 + g\rho_{(1)} h_{(1)} h_{(2)}) = 0. \tag{6.11}$$

There are no other independent conservation laws of type (6.10) — all such laws are represented as linear combinations of conservation laws (6.1) (6.3), (6.7), and (6.11) with constant coefficients. The system of conservation laws is thus still not complete, and we need either to be able to correctly describe discontinuous solutions of the shock-wave type for system (6.1) (6.4) or to prove that no such solutions exist (for example, we need to prove that the problem of the decay of an arbitrary initial discontinuity at t > 0 has a smooth solution). It is clear from (6.2), (6.4) that there are other conservation laws in the unidimensional case (n = 1):

$$\frac{\partial w_{(1)}}{\partial t} + \frac{\partial}{\partial x}[v^2_{(1)}/2 + g(h_{(1)} + h_{(2)})] = 0,$$

$$\frac{\partial w_{(2)}}{\partial t} + \frac{\partial}{\partial x}[v^2_{(2)}/2 + g(\lambda h_{(1)} + h_{(2)})] = 0.$$

Since these laws disappear in the transition to two-dimensional problems, they cannot be used as a basis for solving problems with discontinuities.

7. Allowance for Heat. Let us generalize the construction in Parts 4 and 5. We will assume that there is one independent parameter s (unit entropy), so that the local state of the system is characterized by the set $(\rho_{(1)}, v_{(1)}, \rho_{(2)}, v_{(2)}, s)$. As in Part 4, we find it convenient to change over to u ≡ (ρ, ϰ, s, v, w). The complete set of independent scalars: ρ, ϰ, s, w^2. The tensor classification is obvious. For example, Galilean vector X(u) has the general form: X = a(ρ, ϰ, s, w^2)V + b(ρ, ϰ, s, w^2)W. We again adopt hypothesis I for energy E(u):

$$E \equiv \rho_{(1)} v^2_{(1)}/2 + \rho_{(2)} v^2_{(2)}/2 + \rho\varepsilon_0(\rho, \varkappa, s),$$
$$d\varepsilon_0 = Tds - pd(1/\rho) + \mu_{(1)0} d\varkappa,$$

or

$$E = \rho(\varepsilon + v^2/2), \; \varepsilon = \varepsilon_0 + \varkappa(1 - \varkappa)w^2/2; \tag{7.1}$$

$$d\varepsilon = Tds - pd(1/\rho) + [\mu_{(1)0} + (1 - 2\varkappa)w^2/2]d\varkappa + \varkappa(1 - \varkappa)w^k dw^k. \tag{7.2}$$

The structure of the mass-momentum balance equations is determined as in Part 4, while terms containing ∇s appear in the expression for $F_{(1)}$. Since we have introduced the scalar s, we need to supplement the system with the scalar

conservation law $\mathrm{div}X(u) = \theta(\rho, \varkappa, s, w^2)$, where X(u) is a Galilean vector. In accordance with the above proposition, this equation corresponds physically to the law of entropy conservation, i.e.

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\cdots)^k = \theta.$$

Thus, $X = pV + \zeta_{(12)}(\rho, \varkappa, s, w^2)W$. One consequence of these equations is an exact energy conservation law

$$\frac{\partial E(u)}{\partial t} + \frac{\partial R^k(u)}{\partial x^k} = 0. \tag{7.3}$$

In contrast to (4.10), law (7.3) is exact. It is exact because all forms of energy (including heat) were accounted for in (7.1)-(7.2), while only mechanical forms of energy and work were considered in (4.10). Now, as in Part 4, we find the integrating factors. To do this, we convert (7.2) to complete differential form. The calculations are then performed as in Part 4, which leads to the analog of Eqs. (4.14)-(4.18). For simplicity, we will present only the minimal system. The analog of Eqs. (5.4)-(5.6) has the form

$$\frac{\partial \rho_{(1)}}{\partial t} + \frac{\partial}{\partial x^k}(\rho_{(1)} v^k_{(1)}) = q_{(1)}; \tag{7.4a}$$

$$\frac{\partial}{\partial t}(\rho_{(1)} v^j_{(1)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} v^k_{(1)} v^j_{(1)}) + \varkappa \frac{\partial p}{\partial x^j} + \varkappa(1 - \varkappa)\rho \frac{\partial \mu_{(1)0}}{\partial x^j} + \zeta_{(12)} \frac{\partial T}{\partial x^j} = q_{(1)} v^j - f(v^j_{(1)} - v^j_{(2)}); \tag{7.4b}$$

$$\frac{\partial \rho_{(2)}}{\partial t} + \frac{\partial}{\partial x^k}(\rho_{(2)} v^k_{(2)}) = q_{(2)}; \tag{7.5a}$$

$$\frac{\partial}{\partial t}(\rho_{(2)} v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(2)} v^k_{(2)} v^j_{(2)}) + (1 - \varkappa)\frac{\partial p}{\partial x^j} + \varkappa(1 - \varkappa)\rho \frac{\partial \mu_{(2)0}}{\partial x^j} + \zeta_{(21)} \frac{\partial T}{\partial x^j} = q_{(2)} v^j - f(v^j_{(2)} - v^j_{(1)}); \tag{7.5b}$$

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\rho s v^k + \zeta_{(12)} w^k) = \theta; \tag{7.6}$$

$$\theta \equiv (fw^2 - \mu_{(1)} q_{(1)})/T, \quad \mu_{(1)} \equiv \mu_{(1)0} - (1 - 2\varkappa)w^2/2. \tag{7.7}$$

Here, $q_{(2)} \equiv -q_{(1)}$; $\mu_{(2)0} \equiv -\mu_{(1)0}$; $\zeta_{(21)} \equiv -\zeta_{(12)}$; the functions $q_{(1)}(\rho, \varkappa, s, w^2)$, $f(\rho, \varkappa, s, w^2)$, $\zeta_{(12)}(\rho, \varkappa, s, w^2)$ are assigned. One consequence of (7.4)-(7.7) is the energy conservation law

$$\frac{\partial}{\partial t}(\rho \varepsilon_0 + \rho_{(1)} v^2_{(1)}/2 + \rho_{(2)} v^2_{(2)}/2) + \frac{\partial}{\partial x^k}\{\rho_{(1)} v^k_{(1)}[\varepsilon_0 + v^2_{(1)}/2 + p/\rho + (1 - \varkappa)\mu_{(1)0} + T\zeta_{(12)}/\rho_{(1)}] + \rho_{(2)} v^k_{(2)}[\varepsilon_0 + v^2_{(2)}/2 + p/\rho + \varkappa\mu_{(2)0} + T\zeta_{(21)}/\rho_{(2)}]\} = 0. \tag{7.8}$$

Adding (7.4a) to (7.5a) and adding (7.4b) to (7.5b), we obtain the mass-momentum conservation law for the system as a whole:

$$\frac{\partial}{\partial t}(\rho_{(1)} + \rho_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} v^k_{(1)} + \rho_{(2)} v^k_{(2)}) = 0; \tag{7.9a}$$

$$\frac{\partial}{\partial t}(\rho_{(1)} v^j_{(1)} + \rho_{(2)} v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} v^j_{(1)} v^k_{(1)} + \rho_{(2)} v^j_{(2)} v^k_{(2)} + p\delta^{jk}) = 0. \tag{7.9b}$$

Here, we mention one consideration that helps formulate a hypothesis to determine $\zeta_{(12)}$ in certain situations. Let us assume that part of the entropy $\rho_{(1)} s_{(1)}$ is transported by component 1 and part $\rho_{(2)} s_{(2)}$ is transported by component 2. Thus, $\rho s = \rho_{(1)} s_{(1)} + \rho_{(2)} s_{(2)}$. (For example, in superfluid dynamics, it is assumed that all of the entropy is transported by the normal component.) Then

$$\frac{\partial}{\partial t}(\rho_{(1)} s_{(1)} + \rho_{(2)} s_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} s_{(1)} v^k_{(1)} + \rho_{(2)} s_{(2)} v^k_{(2)}) = \theta.$$

This equation can be written in the following form:

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}[\rho s v^k + \varkappa(1-\varkappa)\rho(s_{(1)} - s_{(2)})w^k] = \theta. \tag{7.10}$$

Comparing (7.6) and (7.10), we see that $\zeta_{(12)} = \varkappa(1-\varkappa)\rho(s_{(1)} - s_{(2)})$.

The equations found here differ both from the equations of two-velocity heterogeneous hydrodynamics described in [11, 12] and from the equations obtained in [5, 6]. The structural hypotheses introduced above are quite natural, but our description of two-velocity physical systems could be refined further by changing from hypothesis I to a more appropriate variant of hypothesis II and/or by introducing additional parameters into the initial set u. No technical problems will arise if these parameters are scalars.

Note. The state of thermodynamic equilibrium is characterized by the relations T = const, p = const, $\mu_{(1)0}$ = const. Deviations from this state lead to a situation in which $\nabla p \neq 0$ and/or $\nabla\mu_{(1)0} \neq 0$ and/or $\nabla T \neq 0$. As can be seen from (7.4b) and (7.5b), forces that will induce motion of the components will be created under such conditions. If the conditions of thermodynamic equilibrium are formulated differently due to the presence of additional factors (gravitation, electromagnetic field, etc.), then corrections must be introduced into the equations.

8. Phase Transformations. In two-velocity thermodynamics, internal energy ε depends on the scalar w^2. Thus, generally speaking, phase transformations are dependent on w^2, and relations of the type $T = T(p, \varkappa)$ which are characteristic of such transformations will be replaced by the relation $T(p, \varkappa, w^2)$ or its analog. New types of phase transitions which are unknown in traditional thermodynamics may also occur. Below, we present the first outline of a general approach to the theory of phase transformations which we believe can also be applied to two-velocity thermodynamics. To do this, let us first return to Part 3 and analyze a van der Waals gas for the sake of illustration.

We take (3.2)-(3.3) as fundamental conservation laws and (3.4) as a closing conservation law that follows from these equations. This structure of the basic equations is associated with the first thermodynamic form (3.5), which includes the canonical potential $\rho\varepsilon = \Phi(\rho, \rho s)$. If potential Φ is determined in the region Ω of variables ρ and ρs (for the sake of simplicity, Ω is assumed to be convex), then in the coordinate space $(\rho, \rho s, \rho\varepsilon)$ we have the manifold M given by the equation $\rho\varepsilon = \Phi(\rho, \rho s)$, $(\rho, \rho s) \in \Omega$. Meanwhile, M is projected one-to-one onto the plane $(\rho, \rho s)$. The conjugate set of variables is (γ, T). The conjugate potential $p = \Phi_*(\gamma, T)$, $(\gamma, T) \in \Omega$ is established if the canonical transformation $(\rho, \rho s) \to (\gamma, T)$ is one-to-one and smooth. Here, Ω_* is the image of Ω in the transformation. We thus obtain a smooth manifold M_* in the conjugate coordinate space (γ, T, p). This manifold can be projected one-to-one onto the plane (γ, T). We additionally establish the smooth transformation π: $M \to M_*$, which is a diffeomorphism of M onto M_*. In this case, Ω will be referred to as the **region of regularity.** No phase transformations take place inside it (without movement to a point outside Ω). Strict convexity of the potential $\Phi(\rho, \rho s)$ is a typical condition of regularity (and local stability) and leads to strict convexity of $\Phi_*(\gamma, T)$. In other words, the absence of phase transformations is interpreted geometrically as the existence of smooth manifold M_* in the conjugate space, this manifold capable of being projected into the region Ω_* in the coordinate space of the conjugate variables (γ, T). **If canonical transformation $(\rho, \rho s) \to (\gamma, T)$ has a singularity, then it is by definition a phase transformation. The exact type of transformation is determined by the type of singularity.** Singularities may occur when the transformation is not smooth (phase transformations of the second or other orders) or if a one-to-one correspondence is lacking (first-order phase transformations). The latter is possible if the potential Φ is not convex at every point. In this case, the manifold M_* is determined by the relations $\gamma = \gamma(\rho, \rho s)$, $T = T(\rho, \rho s)$, $p = p(\rho, \rho s)$, but cannot be projected onto Ω_* (the function $p = p(\gamma, T)$, which gives M_*, is not defined everywhere in Ω_*) and may not be smooth everywhere. Accordingly, the transformation π: $M \to M_*$ has singularities. The analysis of canonical transformation π: $M \to M_*$ from a general standpoint is of central importance in the mathematical theory of phase transformations, and phase transformations can be best classified by classifying the types of singularities that π: $M \to M_*$ can have. As is known, progress in the classification of singularities of differentiable transformations is being made in the study of catastrophe theory.

We take a van der Waals gas:

$$p = \frac{RT}{v-b} - \frac{a}{v^2}, \; c_v = \text{const}. \tag{8.1}$$

Here, v is specific volume; R is the gas constant, dependent on the nature of the gas; a and b are the van der Waals constants; c_V is heat capacity. The critical parameters: $v_k = 3b$, $T_k = 8a/27Rb^2$, $p_k = 27b^2$. We introduce the dimensionless quantities:

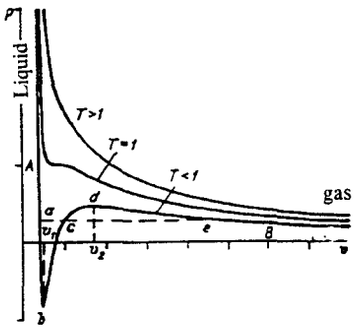

Fig. 1

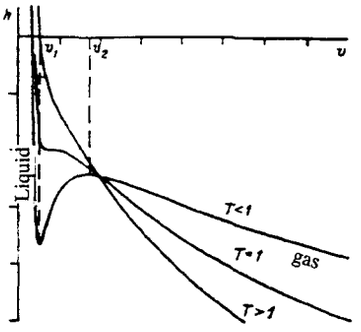

Fig. 2.

$$v' \equiv v/v_k,\ T' \equiv T/T_k,\ p' \equiv p/p_k,$$
$$\varepsilon' \equiv (9b/a)\varepsilon,\ s' \equiv (8/3R)s,\ c'_v \equiv (8/3R)c_V.$$

In dimensionless form, (8.1) becomes

$$p' = \frac{8T'}{3v' - 1} - \frac{3}{(v')^2},\ c'_V = \text{const.} \tag{8.2}$$

The primes will be omitted below to simplify the notation. After performing the necessary calculations, we obtain

$$\gamma - \lambda(T) = T\left\{\frac{8v}{3v-1} - \frac{8}{3}\ln\left(\frac{3v-1}{2}\right)\right\} - \frac{6}{v},\ \lambda(T) \equiv c_V T(1 - \ln T) - Ts_k \tag{8.3}$$

(s_k is the value of entropy s when T = v = 1). We designate $h \equiv \gamma - \lambda$.

The position of a phase transition at T < 1 is usually determined by using Maxwell's rule [13] (Fig. 1). In accordance with the latter, the section of the curve *abcde* (corresponding to the metastable states *ab* and *de* and physically unattainable state *bcd*) is discarded; here, $\gamma(a) = \gamma(e)$. As a result, the equation of state is given by curves *aA* (liquid phase) and *eB* (gas).

We will construct the manifold M_* in the coordinate space (γ, T, p) by using Eqs. (8.2)-(8.3). These equations give the parametric representation $\gamma = \gamma(v, T)$, $p = p(v, T)$ (Figs. 1-3), which is convenient for the given case. The images of points *a*, *e* (Fig. 1) merge to form a single point $a_* = e_*$ (Fig. 3), while curve *abcde* becomes the closed curve $a_*b_*c_*d_*e_*$. In accordance with (3.6), the derivatives at the points b_* and d_* have the form $\partial p/\partial\gamma \mid_{b_*} = 1/v_1 = \rho_{(1)} > 0$, $\partial p/\partial\gamma \mid_{d_*} = 1/v_2 = \rho_{(2)} > 0$ (thus, the derivatives $\partial p/\partial\gamma$ are continuous!). The structure of M_* is now clear. At T > 1, the manifold is smooth and convex. At the critical point T = 1, p = 1, it has a singularity — the manifold becomes convolute and forms a tube corresponding to metastable and physically impossible states. The use of Maxwell's rule is equivalent to removing this tube. Ultimately, the curve $A_*a_*b_*c_*d_*e_*B_*$ becomes a **continuous, strictly convex curve $A_*a_*B_*$ having a discontinuous derivative (point of inflection) at point a_*. Accordingly, M_* is replaced by a strictly convex manifold M^0_* which is smooth everywhere except for a line that begins at the critical point and determines the position of the phase transformation.** It should be noted that that strict convexity of the potential is of decisive importance for proper formulation of the problem of the decay of an arbitrary discontinuity, since it simultaneously provides a **method of regularizing this problem for nonconvex equations of state**. In the regularization of the manifold, the lowest value of γ, i.e. min $\{\gamma_\nu\}$, is chosen within the region of ambiguity, i.e. the region in which certain (p, T) correspond to a set $\{\gamma_\nu\}$ (this condition is equivalent to Gibbs' condition of the stability of thermodynamic equilibrium). In other words, the regularization entails **choosing the largest** p, i.e. max $\{p_\alpha\}$, where certain (γ, T) correspond to a set $\{p_\alpha\}$. The latter means that a Legendre transformation is replaced by a generalized Legendre transformation within the region of ambiguity $\Phi_* \equiv \max(\rho\gamma + \rho sT - \rho\varepsilon)$ (also see [14]). This procedure consists of the following.

1. Starting with the structure of the fundamental and closing laws of conservation (or, more generally, the balance equations) for a suitable set of extensive physical quantities, we establish canonical first-order thermodynamic variables $a = (a_k)$ and the first thermodynamic potential $\Phi(a)$.

2. We construct the first fundamental differential form $d\Phi = A^k da_k$.

3. We find the conjugate thermodynamic variables $A(a) = (A^k(a))$ and conjugate potential $\Phi_* \equiv A^k a_k - \Phi$. Thus, manifold M_* can be given by the parametric equations $Ak = A^k(a)$, $\Phi_* = \Phi_*(a)$ in the conjugate coordinate state $((A^k), \Phi_*)$.

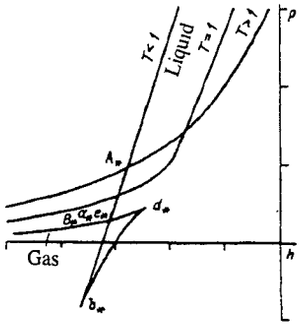

Fig. 3

4. We examine the canonical transformation $a \to A(a)$.

5. If this transformation is not entirely one-to-one, then there will be $A(a)$ that correspond to the set $\{\Phi_{*\alpha}(a)\}$. In this case, the classical Legendre transform is replaced by a generalized transform. We take the expression $\Phi_* \equiv \max_a(A^* a_k - \Phi)$, if the condition of stability of thermodynamic equilibrium $\Phi(a)$ is convex and $\Phi_* \equiv \min_a(A^k a_k - \Phi)$ if it is concave.

Note. In more detailed theories accounting for surface tension, etc, the metastable parts a_*b_* and e_*d_* are also of interest. In fact, if (for example) the system is at point ζ lying on a_*b_*, then a gaseous component is formed from the liquid. In a first approximation, the rate of formation of the new phase for given p and T will be determined by the quantity $q \sim \gamma(a_*) - \gamma(\zeta)$. Similar considerations make it possible to specify $q_{(i)}$ in Eqs. (7.4a), (7.5a) or their analogs.

9. Hydrodynamics of a Superfluid. The first instance of the use of two-velocity hydrodynamics was in the superfluid hydrodynamics constructed by Landau [4]. We will describe and briefly comment on the main postulates of this theory in light of the above. We will assign the normal component the index 1 and the superfluid the index 2.

1. The first hypothesis is that $\rho_{(1)}$ and $\rho_{(2)}$ depend only on temperature, so that $\rho = \rho(T)$ and $\varkappa = \varkappa(T)$.
2. The following expression (in our notation) is used for E

$$E \equiv \rho v^2/2 - \varkappa^2 \rho w^2/2 + E_0(\rho, s, j_0^2),\ j_0 \equiv \varkappa \rho w,$$

It is understood that

$$dE_0 \equiv \Phi d\rho + T d(\rho s) + w^k dj_0^k,$$

where the functions $\Phi(\rho, s, j_0)$ are to be determined. The above expression can also be written in the form

$$\begin{aligned} &E = \rho(\varepsilon + v^2/2),\ \varepsilon = \varepsilon(\rho, s, j_0^2), \\ &d\varepsilon = -\tilde{p} d(1/\rho) + T ds + [(1 - \varkappa)/\rho] w^k dj_0^k. \end{aligned} \tag{9.1}$$

By virtue of hypothesis I, internal energy ε depends explicitly not on the set $(\rho, \varkappa, s, w^2)$ of independent scalars but on a smaller set (ρ, s, J_0^2). On the other hand, the difference between the last terms in (9.1) and (7.2) indicates that, in the general case, some variant of hypothesis II must be used in place of hypothesis I. If the relation $\rho = \rho(T)$ is reversible and if $T = T(\rho)$, $\varkappa = \kappa(\rho)$, then (9.1) can be written in the form

$$\begin{aligned} &d\varepsilon = T ds - p' d(1/\rho) + \varkappa(1 - \varkappa) w^k dw^k, \\ &p' \equiv \tilde{p} + [(1 - \varkappa)\rho d(\rho\varkappa)/d\rho] w^2. \end{aligned} \tag{9.2}$$

Relation (9.2) coincides with (7.2) if ε in (7.2) is independent of $\varkappa$ (and of the independent variable), thus $\mu'_{(1)} = 0$.

3. The next hypothesis states that entropy is transported only by the normal component, i.e. $s_{(2)} = 0$ in (7.10). Thus, $\zeta_{(12)} = (1 - \varkappa)\rho s$.

4. The fourth (and very powerful) hypothesis consists of the expression

$$a_{(2)} \equiv \mathrm{rot} w_{(2)} = 0,\ \frac{\partial w_{(2)}}{\partial t} + \frac{\partial}{\partial x}\left(Q + \frac{v_{(2)}^2}{2}\right) = 0, \tag{9.3}$$

where $Q = Q(\rho, s, J_0^2)$ still must be found.

5. As in the present investigation, the energy conservation law is exact.
6. Finally, Galilean invariance is required of the equations.

The above hypotheses are used to obtain the law of conservation of total momentum

$$\frac{\partial}{\partial t}(\rho_{(1)}v^i_{(1)} + \rho_{(2)}v^i_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)}v^i_{(1)}v^k_{(1)} + \rho_{(2)}v^i_{(2)}v^k_{(2)} + G\delta^{ik}) = 0,$$
$$G \equiv \tilde{p} + \varkappa(1 - \varkappa)\rho w^2. \tag{9.4}$$

If $T = T(\rho)$, i.e. if (9.2) is valid, then $G = p' - [\rho(1 - \varkappa)d\varkappa/d\rho]w^2$. In [4], G was referred to as pressure and designated as p (compare this with (4.15), however). Since $\varkappa$ is not an independent scalar, there will be no separate equation analogous to (5.2a) for $\varkappa$.

In classical hydrodynamics, the Lamb–Gromeka equation [15] is valid

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{v^2}{2}\right) = v \times \omega. \tag{9.5}$$

If passage to the limit for $\varkappa \to 1$ were possible in in the given hydrodynamic theory, then (9.5) would be obtained in the limit from (9.3). Alternatively, with allowance for $\omega_{(2)} = 0$, we would obtain the equality $(1/\rho)\partial p/\partial x^j = \partial Q/\partial x^j$. The expression $(1/\rho)\partial p/\partial x^j$ can be represented as the gradient of the scalar Q only in very special cases: $\rho = \text{const}$ or $p = p(\rho)$, or in the limit at $T \to 0$ with suitable degeneration of $\rho(T)$. In other words, except for special cases, the state $\varkappa \approx 1$ is impossible in the hydrodynamic theory in question.

Thus, compared to the theory explained in Part 7, here we take a different differential form for internal energy ε and adopt very powerful hypotheses 1 and 4. Hypothesis 2 is probably invalid, despite the arguments in [4].

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